11/22

2021年11月22日　　午前 8:03

Surfaces and Calculus

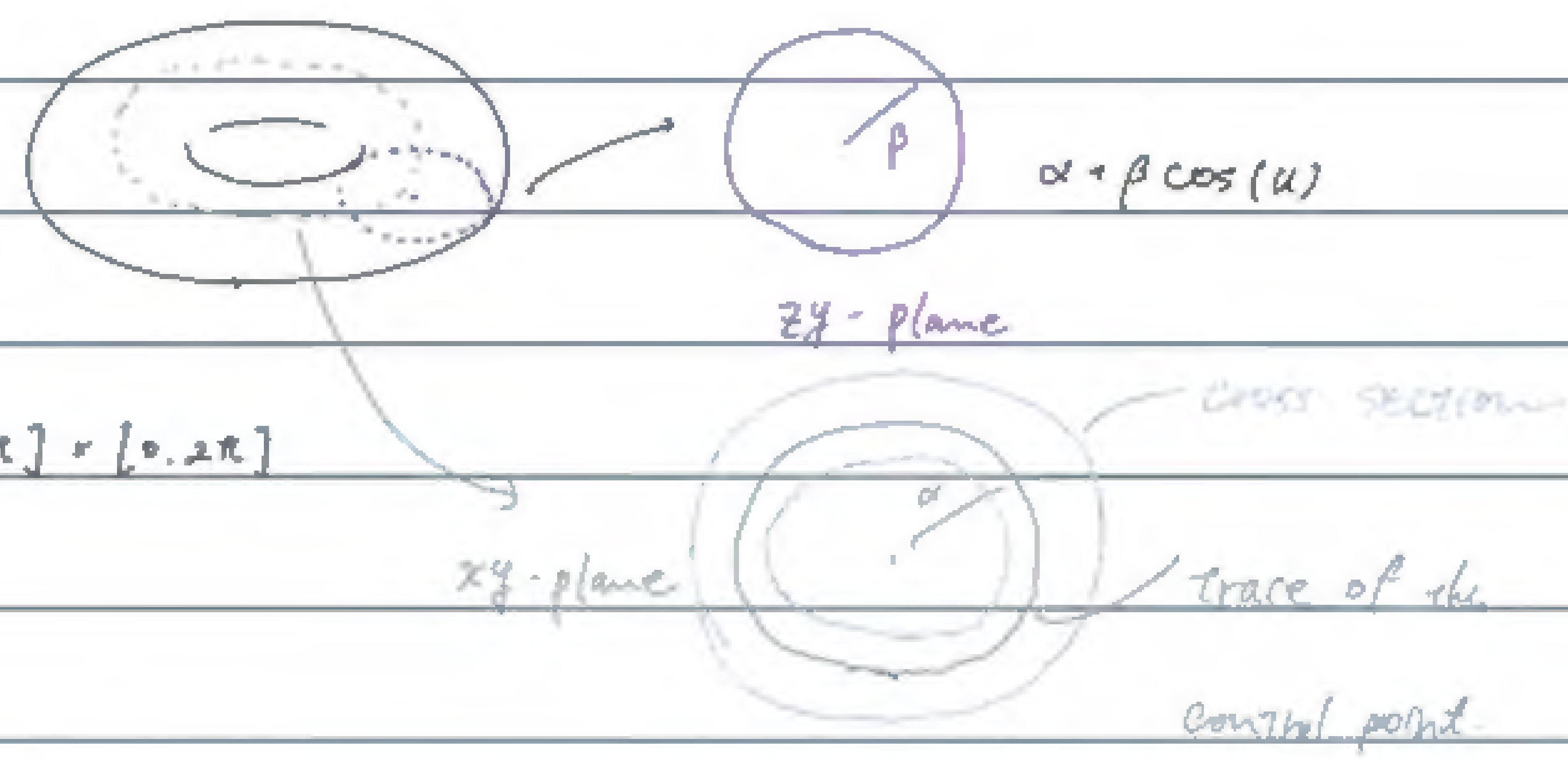
Last time: a surface in $\mathbb{R}^3$ has the form $\vec{S}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

on some domain $D \subseteq \mathbb{R}^2$

Ex. The torus w/ major radius $\alpha \geq 0$ and minor radius β

(w/ $\alpha > \beta > 0$) is the surface

$$\vec{S}(u,v) = \langle (\alpha + \beta\cos(u))\cos(v), (\alpha+\beta\cos(u))\sin(v), \beta\sin(v)\rangle$$



$D: [0, 2\pi] \times [0, 2\pi]$

I. Tangent planes

The tangent plane to surface $\vec{S}(u,v)$ at point $(a,b) \in D$

has normal vector $\vec{n}(a,b) = \vec{S}_u(a,b) \times \vec{S}_v(a,b)$

Note. $\vec{S}_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$ can also be written $\frac{\partial S}{\partial u}$.

Ex. Consider the torus $\vec{S}(u,v)$ w/ major radius 10 and minor radius 5.

What's the tangent plane to $S(u,v)$ at $(\frac{\pi}{4}, \frac{3}{4}\pi)$?

Ans. $\vec{S}(u,v) = \langle (10+5\cos(u))\cos(v), (10+5\cos(u))\sin(v), 5\sin(u)\rangle$

$\vec{S}_u = \langle -5\sin(u)\cos(v), -5\sin(u)\sin(v), 5\cos(u)\rangle$

$\vec{S}_v = \langle -(10+5\cos(u))\sin(v), (10+5\cos(u))\cos(v), 0\rangle$

$\vec{n}(u,v) = S_u(u,v) \times S_v(u,v)$

$$= \begin{vmatrix} i & j & k \\ -5\sin u\cos v & -5\sin u\sin v & 5\cos u \\ -(10+5\cos u)\sin v & (10+5\cos(u))\cos(v) & 0 \end{vmatrix}$$

$$\begin{vmatrix} -5\sin u\cos v & -5\sin u \sin v & 5\cos u \\ -(10+5\cos u)\sin v & (10+5\cos(u))\cos(v) & 0 \end{vmatrix}$$

$$= -5\,(10+5\cos(u))\,\langle \cos(u)\cos(v),\ \cos(u)\sin(v),\ \sin(u)\rangle$$

└ at every $(u,v) \in \mathrm{dom}(\vec{S})$, this is the normal vector at $S(u,v)$

Now at the point

$$\vec{S}\left(\tfrac{\pi}{4}, \tfrac{3}{4}\pi\right) = \left\langle \left(10+5\cos\left(\tfrac{\pi}{4}\right)\right)\cos\left(\tfrac{3}{4}\pi\right),\ \left(10+5\cos\left(\tfrac{\pi}{4}\right)\right)\sin\tfrac{3}{4}\pi,\ \sin\tfrac{\pi}{4}\right\rangle$$

$$= \left\langle -\tfrac{10}{\sqrt{2}} - \tfrac{5}{2},\ \tfrac{10}{\sqrt{2}} + \tfrac{5}{2},\ \tfrac{1}{\sqrt{2}}\right\rangle$$

we have normal vector

$$\vec{n}\left(\tfrac{\pi}{4}, \tfrac{3}{4}\pi\right) = -5\left(10+\tfrac{5}{\sqrt{2}}\right)\left\langle \tfrac{1}{\sqrt{2}}\cdot -\tfrac{1}{\sqrt{2}},\ \tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{2}},\ \tfrac{1}{\sqrt{2}}\right\rangle$$

$$= -25\left(2+\tfrac{1}{\sqrt{2}}\right)\left\langle -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{\sqrt{2}}\right\rangle$$

$\therefore$ The tangent plane at this point is given by

$$\vec{n}\cdot(\vec{x}-\vec{p}) = 0$$

i.e. $\vec{n}\left(\tfrac{\pi}{4}, \tfrac{3}{4}\pi\right)\cdot\left(\vec{x} - \vec{S}\left(\tfrac{\pi}{4}, \tfrac{3}{4}\pi\right)\right) = 0$

i.e. $-25\left(2+\tfrac{1}{\sqrt{2}}\right)\left\langle -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{\sqrt{2}}\right\rangle \cdot \left\langle x + \tfrac{10}{\sqrt{2}} + \tfrac{5}{2},\ y - \tfrac{10}{\sqrt{2}} - \tfrac{5}{2},\ z - \tfrac{1}{\sqrt{2}}\right\rangle$

i.e. $-\tfrac{1}{2}\left(x + \tfrac{10}{\sqrt{2}} + \tfrac{5}{2}\right) + \tfrac{1}{2}\left(y - \tfrac{10}{\sqrt{2}} - \tfrac{5}{2}\right) + \tfrac{1}{\sqrt{2}}\left(z - \tfrac{1}{\sqrt{2}}\right) = 0.$

II. Surface Area

The surface area of a surface $\vec{S}(u,v)$ parameterised on domain D is

$$A = \iint_D |\vec{S}_u \times \vec{S}_v|\, dA$$

Q: where is the formula coming from?

A: Piecewise approximation of Surface S via parallelograms.

limiting these approximations yields that formula

Note: for this to work, we assume that $\vec{S}(u,v)$

surface once on D.

Ex. Compute the surface area of the torus w/ major radius 10 & minor 5.

Ans. $\vec{n}(u,v) = \vec{S}_u(u,v) \times \vec{S}_v(u,v) = -5\left((10+5\cos(u)\right) \langle \cos(u)\cos(v), \cos(u)\sin(v), \sin(u) \rangle$

$$\left|\vec{S}_u(u,v) \times \vec{S}_v(u,v)\right| = \left|-5\left(10+5\cos(u)\right)\right| \sqrt{\cos^2(u)\cos^2(v) + \cos^2(u)\sin^2(v) + \sin^2(u)}$$

$$= 25\,|2+\cos(u)| \sqrt{\cos^2(u)\left(\cos^2(v)+\sin^2(v)\right) + \sin^2(u)}$$

$$= 25\left(2+\cos(u)\right)$$

$$\therefore \text{Area}(S) = \iint_{D(S)} |\vec{S}_u \times \vec{S}_v|\, dA$$

$$= \int_{u=0}^{2\pi}\int_{v=0}^{2\pi} 25\left(2+\cos(u)\right) dv\, du$$

$$= 200\pi^2$$

Note: If $f(x,y)$ is a function, the graph is a surface $\vec{S}(x,y) = \langle x, y, f(x,y) \rangle$. The normal vector to this surface is

$$\vec{n}(x,y) = \vec{S}_x \times \vec{S}_y$$

$$= \det \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$= \langle -f_x, -f_y, 1 \rangle$$

$$\therefore \text{Area}(\text{graph } f) = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

Idea: Surface area is an area. So we should be able to write

$$\text{Area}(S) = \iint_S 1\, dS$$

resembles formula $\text{Area}(R) = \iint_R 1\, dA$

To make this analogy work, $dS = \underbrace{|\vec{S}_u \times \vec{S}_v|}_{\text{Jacobian}}\, dA$

III. Surface Integrals

The integral of function $f(x,y,z)$ over surface S parameterized by $\vec{S}(u,v)$ on D is

by $\vec{S}(u,v)$ on D is

$$\iint_S f\,dS = \iint_D f(S(u,v))\,|\vec{S}_u \times \vec{S}_v|\,dA$$